An. Şt. Univ. Ovidius Constanţa Vol. 3(1), 1995, 63–82

SOLUTION OF 3D ELLIPTIC SYSTEMS BY SEMI-REFINEMENT

P. W. Hemker

Abstract

To numerically solve elliptic partial differential equations in three space dimensions, these equations are discretised and huge systems of (linear or nonlinear) equations arise. To solve these equations, multigrid methods are the most efficient technique and they solve the systems with $\mathcal{O}(N)$ arithmetic operations, where N is the number of degrees of freedom in the discretisation. In a previous paper we have shown how semi-refinement can be used to construct multigrid methods for 3 dimensions. Adaptive approximation techniques, based on semi-refinement, can be used to minimise the number N for a given accuracy. For smooth solutions, such techniques automatically lead to sparse grids over the domain of definition.

After a general introduction, in this paper we analyse the accuracy of low order piecewise polynomial approximation on regular or sparse grids, in different norms.

1 Introduction

The basic model problem to demonstrate the value of numerical methods for general elliptic boundary value problems has always been

$$-\Delta u = f \;\text{ in }\; \Omega = (0,1)^2 \quad u = 0 \;\text{ on }\; \partial\Omega\,. \tag{1}$$

If a uniform $n \times n$-mesh is placed over Ω, i.e. that $n+1$ equidistant mesh-lines are drawn in the horizontal, and the same number in the vertical direction, the distance between the mesh-lines is called the mesh-width, $h = 1/n$. The

Received by the editors: June,1995

grid points are x_{ij}, where $0 \leq i,j \leq n$. If we want to approximate the solution of (1) numerically, discretisation is applied to get a set of linear equations:

$$AU = F\,, \tag{2}$$

where, typically, $F_{(i-1)(n+1)+j} = h^2 f(x_{ij})$, and A is a matrix with a special structure. The system has $N = (n+1)^2$ equations and the same number of unknowns.

The element $U_{(i-1)(n+1)+j}$ of the solution vector U in the system of equations (2) represents the approximate solution of equation (1) at the point x_{ij}, i.e. $U_{(i-1)(n+1)+j} \approx u(x_{ij})$. The accuracy of this approximation depends on the type of discretisation and on h, the width of the mesh applied. This means that the approximation becomes more accurate if more mesh-lines are introduced. Typically, for a simple discretisation method, the error in the solution, $|U_{(i-1)(n+1)+j} - u(x_{ij})|$, is proportional to h^2.If higher accuracies are required, smaller values of h are needed, i.e. a large number of mesh-points are necessary. Such large numbers of mesh-points give rise to very large systems (2), and the techniques used to solve such systems of moderate size (e.g., Gauss elimination) cannot be applied because the number of arithmetic operations (the number of additions and multiplications) to compute the solution by these methods is proportional to N^3.

For large systems of type (2), Gauss elimination would take too much time, even on present day's fastest computers, and different methods are used, that take advantage of the special properties of such equations. All these special methods to solve discretised PDEs are iterative methods, where a first guess of the solution is improved step by step in an iteration process. Until the sixties, simple relaxation methods were very popular. Here, all separate equations in (2) are scanned one by one, and each time when an equation is visited, the corresponding unknown is updated, based on the present information about the other unknowns.

Later, in the seventies, more efficient iterative methods, based on the construction of Krylov spaces, appeared, such as the preconditioned conjugate gradient method, GMRES or, a more recent development, BiCGStab. Nowadays, these methods are the most popular ones to solve the very large systems. One reason is that these methods are relatively easy to implement in a computer program.

However, to restrict the amount of work to $\mathcal{O}(N)$, we have to resort to multigrid (MG) methods. These methods have a more complex structure. Invented in the sixties, they got the full attention of the numerical community not before 1980. A pioneering paper in the late seventies [1] started the interest, and at present the multigrid method is well-accepted and it is successfully applied [3] in many fields.

Multigrid and semi-coarsening

The principle behind the MG technique is the fact that simple relaxation techniques only efficiently reduce the high-frequency errors on a mesh, and that the low-frequency errors can better be reduced by a discrete equation on a related coarser mesh which contains essentially less mesh points. Now the MG method uses this principle recursively to solve the problem on the coarser meshes (see Figure (1). All computational work together (on the coarse and the fine meshes) to solve the differential problem as accurate as is possible on the finest mesh (with N mesh points) is still $\mathcal{O}(N)$.

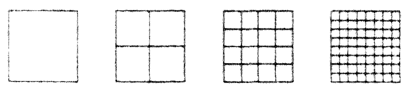


Figure 1: A classical sequence of grids in two dimensions.

It is well known how multigrid methods can be used for two-dimensional (2D) problems, and that the same techniques can be used for three-dimensional (3D) problems as well. One may even point to the fact that the total amount of work on the coarse grids is relatively smaller in the 3D-case than in the 2D-case. However, the reverse side is that only a relatively small amount of error components can be annihilated by these coarse grid corrections. The consequence is that in the 3D-case powerful relaxation methods are required to reduce the total error with a sufficient efficiency.

E.g., one such relaxation procedure is alternating plane-relaxation, in which all planes in the cube are visited by different orderings, and where for each plane a 2D sub-problem is solved (by a 2D MG method). This procedure is not very attractive, because there are many possibilities to order the planes in the cube, and a choice has to be made by what ordering the planes have to be visited. For a general problem such a choice is artificial, and the one choice may be better for the one problem while another choice can be advantageous in another situation. Such 3D-methods are also hard to vectorise or to parallelise so that we may have little advantage of new computer architectures.

However, there exists an alternative [2]. Already for 2D fluid flow problems it has become clear that it is sometimes better to generate coarser grids, not by taking together a 2×2 set of four small cells to form one bigger cell, but to take together only 2 cells, so that a coarser mesh is obtained with a different mesh-size ratio. This is the principle of ***semi-coarsening***. Here also we have the argument that the semi-coarsening is direction-dependent, and that there are more ways to assemble pairs of cells to form the coarse grids. But in the general, problem-independent case we may apply both semi-coarsenings at the same time. In that case the fine grid has two corresponding coarse grids. Now we have to study how the corrections from both coarse grids can cooperate to yield a good coarse grid correction for the solution on the finer grid.

We can approach the same technique from the other side. We may start with a coarse grid and make finer and finer grids, each time by halving grid cells into two finer cells (see Figure 2). This principle of refinement can also be applied in three dimensions. In this case the number of possible grid refinements is even larger (see Figure 4).

This approach of semi-refinement can be very powerful when combined with adaptive meshing, i.e., in all meshes only those cells are created that really contribute best to the reduction of the total error. Here the idea of hierarchical basis plays an important role, in order to combine the function approximations on the different grids into a single, unique representation.

In the following section we first introduce the notational framework to allow a technical discussion of the problems involved. We introduce the multidimensional multiresolution analysis, and more-dimensional wavelet spaces, which are the right tools to introduce hierarchical bases for regular and sparse grid approximations. In the next section we describe piecewise constant and piecewise linear approximation in more dimensions, and we give errorbounds for these approximations on regular and on sparse grids.

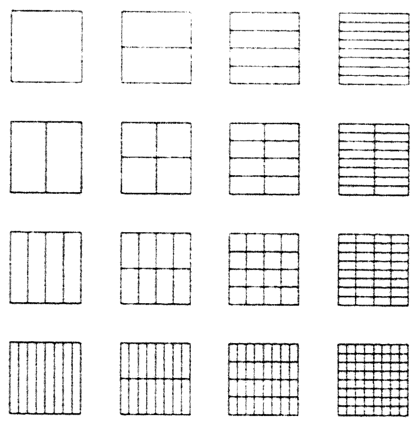


Figure 2: A family of semi-refined grids in two dimensions.

the 2^{n-1}-dimensional subspace in V_n of functions vanishing at the nodes of the V_{n-1}-mesh. The corresponding spaces of functions that satisfy homogeneous Dirichlet boundary conditions are denoted by $V_n^0 \subset V_n$ and $W_n^0 \subset W_n$. Notice that in the one-dimensional case $W_n^0 = W_n$.

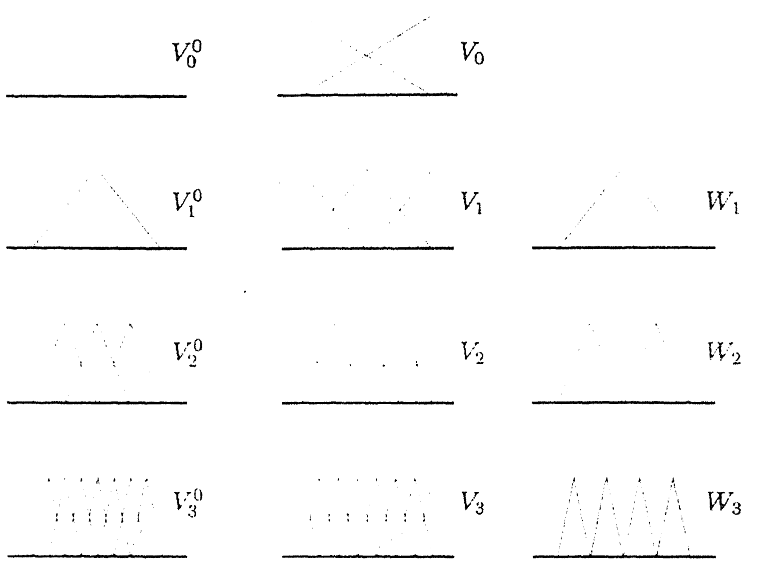


Figure 3: Basis functions on the interval $[0,1]$ in the spaces V_n^0, V_n and W_n.

For $\Omega = \mathbb{R}$ a similar sequence $\{V_n\}_{n=0,1,2,\ldots}$ can be constructed, with $h_n = 2^{-n}$, and this sequence can be completed in the natural way with $\{V_n\}_{n=-1,-2,\ldots}$ Now we have formed for $L^2(\mathbb{R})$ a *multiresolution analysis* $\{V_n\}_{n\in\mathbb{Z}}$.

By R_j we denote a projection $R_j : X(\Omega) \to V_j$, such that for $u \in X(\Omega) \cap C(\Omega)$

$$R_j u \in V_j \quad \text{and} \quad (R_j u)(x_i) = u(x_i) \quad \forall x_i \in \Omega_j^+.$$

For a given function $f \in X(\Omega)$, the "difference information" between two successive approximations $R_j f \in V_j$ and $R_{j-1} f \in V_{j-1}$ is given by the projection $Q_j f$ of f onto the complement W_j of V_{j-1} in V_j,

$$\begin{aligned} V_{j-1} \oplus W_j &= V_j, \\ V_{j-1} \cap W_j &= \{0\}, \\ Q_j f &= R_j f - R_{j-1} f. \end{aligned}$$

The four requirements (3.1) to (3.4) imply that the spaces W_j are also scaled

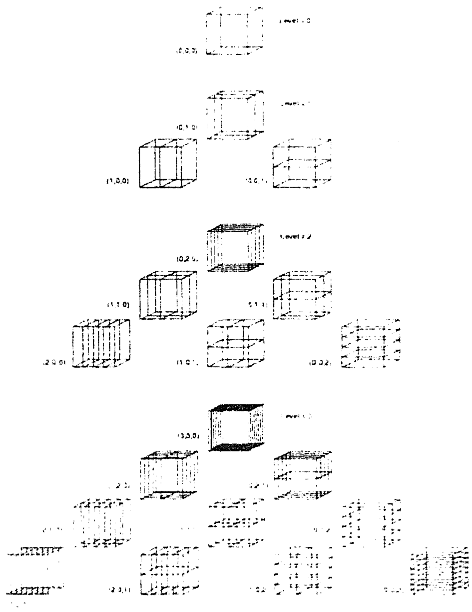


Figure 4: Semi-refinement of the cube.
Grids on levels 0, 1, 2 en 3.

versions of one space W_0,

$$f(x) \in W_j \Leftrightarrow f(2^{-j}x) \in W_0, \tag{9}$$

that they are translation invariant for the discrete translations $2^{-j}\mathbb{Z}$,

$$f(x) \in W_j \Leftrightarrow f(x - 2^{-j}k) \in W_0, \tag{10}$$

and that $\{W_j\}$ are mutually linearly independent spaces, generating all of $X(\Omega)$, cf. (7). For $\Omega = [0,1]$ and homogeneous boundary conditions, we know $V_0^0 = \{0\}$ and hence

$$V_n^0 = \bigoplus_{j=1}^{n} W_j.$$

As soon as we find a function $\psi(x)$ with the property that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ is a basis of W_0, then by a simple rescaling we see that $\{\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$, is a basis of W_j. Since $X(\Omega)$ is the direct sum of these W_j, the full collection $\{\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ is a *hierarchical basis* for $X(\Omega)$.

The first approximation of an arbitrary function from $L^2(\mathbb{R})$ consists in writing $f(x) = \sum f_j(x)$, where each f_j belongs to the corresponding *channel* W_j. Typically, we write for $n > m$,

$$V_n = V_m \oplus W_{m+1} \oplus W_{m+2} \oplus \cdots \oplus W_n .$$

In this way we obtain a decomposition of the function f in channels, and, by taking more spaces W_n, we get a larger sequence of ***hierarchical approximations*** of f.

In the one-dimensional case, both for $\Omega = [0,1]$ and for $\Omega = \mathbb{R}$, each W_j has its natural basis, the *standard basis* $\{\psi_k^j\}$ consisting of basis functions ψ_k^j with minimal support. These piecewise linear basis functions ψ_k^j may be characterised either by their support $2^{1-j}[k, k+1]$ or by their center points $x_k^j = 2^{1-j}(k + \frac{1}{2})$.

In fact, the family of piecewise linear basis functions $\{\psi_k^j\}_{0 \le k < 2^j, 0 \le j < n}$, or $\{\psi_k^j\}_{k \in \mathbb{Z}, j < n}$ (for $\Omega = [0,1]$ or $\Omega = \mathbb{R}$ respectively) forms a hierarchical basis for $f \in V_n$, and with

$$\psi(x) = \begin{cases} x & \text{if } x \in [0,1], \\ 2 - x & \text{if } x \in [1,2], \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$f(x) \approx \sum_{j,k} a_{jk} \psi_k^j(x) = \sum_j \sum_k a_{jk} \psi(2^j x - k). \tag{11}$$

Pw linear approximation in d dimensions

We approximate $u \in X = C^0(\Omega)$ by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, in the space of piecewise d-linear functions on $\Omega_{\boldsymbol{n}}$, i.e.

$$V_{\boldsymbol{n}} = \text{Span}\{\phi_{\boldsymbol{nj}}\}, \tag{12}$$

with, for some $q \ge 1$ or $q = \infty$,

$$\begin{aligned} &\phi_{\boldsymbol{nj}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q} \phi(2^{\boldsymbol{n}} \boldsymbol{x} - \boldsymbol{j}), \\ &\phi(\boldsymbol{x}) = \textstyle\prod_{j=1}^{d} \Lambda(x_j), \\ &\text{with } \Lambda(x) = \max(0, 1 - |x|), \text{ the usual } \textit{hat function}. \end{aligned} \tag{13}$$

We define the projection

$$\begin{aligned} R_{\boldsymbol{n}} :\ & X \to V_{\boldsymbol{n}} \subset X \\ & u \mapsto u_{\boldsymbol{n}} = R_{\boldsymbol{n}} u, \\ & u_{\boldsymbol{n}}(\boldsymbol{x}) = u(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega_{\boldsymbol{n}}^+ . \end{aligned} \tag{14}$$

We notice that the operator $R_{\boldsymbol{n}} = R_{n_1,\cdots,n_d}$ can be decomposed as

$$R_{\boldsymbol{n}} = \prod_{j=1}^{d} S^1_{h_j \boldsymbol{e}_j}, \quad \text{with } \boldsymbol{h} = \boldsymbol{h}_{\boldsymbol{n}} = (2^{-n_1}, \cdots 2^{-n_d}), \tag{15}$$

where $S^1_{h_j \boldsymbol{e}_j} u(\boldsymbol{x})$ is a function, piecewise linear in the j-th coordinate direction, such that $S^1_{h_j \boldsymbol{e}_j} u(\boldsymbol{x}) = u(\boldsymbol{x})$ for all $\boldsymbol{x}$ with $x_j/h \in \mathbb{Z}$.

Pw constant approximation in d dimensions

We approximate $u \in X = L_p^{\mathrm{loc}}(\Omega)$ by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, in the space of piecewise constant functions on $\Omega_{\boldsymbol{n}}$, i.e.

$$V_{\boldsymbol{n}} = \mathrm{Span}\,\{\phi_{\boldsymbol{n}\boldsymbol{j}}\}, \tag{16}$$

with, for some $q \geq 1$ or $q = \infty$,

$$\begin{aligned} &\phi_{\boldsymbol{n}\boldsymbol{j}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q} \phi(2^{\boldsymbol{n}} \boldsymbol{x} - \boldsymbol{j}), \\ &\phi(\boldsymbol{x}) = \textstyle\prod_{j=1}^{d} \chi_{[0,1]}(x_j), \\ &\text{with } \chi_{[0,1]}(x) \text{ the } \textit{characteristic function} \text{ on the unit interval.} \end{aligned} \tag{17}$$

We define the projection

$$\begin{aligned} R_{\boldsymbol{n}} :\quad & X \to V_{\boldsymbol{n}} \subset X \\ & u \mapsto u_{\boldsymbol{n}} = R_{\boldsymbol{n}} u, \text{ with} \\ & u_{\boldsymbol{n},\boldsymbol{i}} = u_{\boldsymbol{n}}((\boldsymbol{i} + \boldsymbol{e}/2)\boldsymbol{h}) = 2^{|\boldsymbol{n}|} \textstyle\int_{\Omega_{\boldsymbol{n}\boldsymbol{i}}} u(\xi)\, d\Omega\,. \end{aligned} \tag{18}$$

We notice that the operator $R_{\boldsymbol{n}} = R_{n_1,\cdots,n_d}$ can be decomposed as

$$R_{\boldsymbol{n}} = \prod_{j=1}^{d} S^0_{h_j \boldsymbol{e}_j}, \quad \text{with } \boldsymbol{h} = \boldsymbol{h}_{\boldsymbol{n}} = (2^{-n_1}, \cdots 2^{-n_d}), \tag{19}$$

where $S^0_{h_j \boldsymbol{e}_j} u(\boldsymbol{x})$ is a function, piecewise constant in the j-th coordinate direction, such that

$$S^0_{h\boldsymbol{e}_j} u(\boldsymbol{x}) = \frac{1}{h} \int_{\boldsymbol{x}-(h/2)\boldsymbol{e}_j}^{\boldsymbol{x}+(h/2)\boldsymbol{e}_j} u(x_1, \cdots, x_d)\, dx_j$$

for all $\boldsymbol{x}$ with $(x_j/h \pm 1/2) \in \mathbb{Z}$.

If we take $p = 2$, then $X = L^2(\Omega)$ is a Hilbert space, and $\{\phi_{\boldsymbol{n}\boldsymbol{j}}\}$ is an orthonormal basis in $V_{\boldsymbol{n}}$. Moreover, $\mathbb{R}_{\boldsymbol{n}}$ is the orthogonal projection $L^2(\Omega) \to V_{\boldsymbol{n}}$.

For $\Omega = \mathbb{R}^d$, the set $\{V_{\boldsymbol{n}}\}$ as defined in (16)-(17) is a typical multiresolution analysis. This is no longer the case if we consider a bounded domain Ω. Nevertheless much of the decomposition procedures still can be used.

Sparse grid approximation

As shown above, with the spaces $V_{\boldsymbol{n}}$ we can associate the wavelet spaces $W_{\boldsymbol{k}}$ and a hierarchical basis $\{\psi_{\boldsymbol{k},\boldsymbol{j}}\}$ to approximate functions in $X(\Omega)$. We shall see in the next section that the contribution of the $\psi_{\boldsymbol{k},\boldsymbol{j}}$-component to the error reduction is

$$2^{-d}|u|\,\|\boldsymbol{h}\|^2 = 2^{-d}|u|\,|\text{support}(\psi_{\boldsymbol{k},\boldsymbol{j}})|^2.$$

To optimally reduce the error for the least number of degrees of freedom, we should add degrees of freedom $(\boldsymbol{j},\boldsymbol{k})$ with the magnitude of the quantity $|u|\,|\text{support}(\psi_{\boldsymbol{k},\boldsymbol{j}})|^2$ as criterion. E.g., for an equally distributed $\frac{\partial^4 u}{\partial^2 x \partial^2 y}$, to obtain an optimal efficiency we should construct the discrete space as

$$\text{span}\{\psi_{\boldsymbol{k},\boldsymbol{j}}|\ \text{support}(\psi_{\boldsymbol{k},\boldsymbol{j}}) \leq C\}.$$

This leads to the sparse grid as introduced by Zenger [4].

Definition 3.1 A *sparse grid approximation space* is the space

$$\hat{V}_n(\Omega) = V_{\boldsymbol{0}} \oplus \bigoplus_{|\boldsymbol{n}|<n;\boldsymbol{n}\geq\boldsymbol{0}} W_{\boldsymbol{n}}(\Omega).$$

The center points of the supports of the natural basis functions in $\hat{V}_n$ form the *sparse grid* $\hat{\Omega}_n^*$.

Definition 3.2 We define the *sparse grid approximation operator*, $\hat{R}_n$, as follows: $\hat{R}_n u$ is the interpolant of u on the sparse grid $\hat{\Omega}_n^*$ in $\hat{V}_n$, i.e.

$$\hat{R}_n u \in \hat{V}_n: \quad (\hat{R}_n u)(x_i) = u(x_i) \qquad \forall x_i \in \hat{\Omega}_n^*.$$

4 Error estimates

We approximate $u \in C^{1,1,1}(\Omega)$ by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, where $V_{\boldsymbol{n}}$ denotes the space of piecewise constant functions on $\Omega_{\boldsymbol{n}}$. We can write

$$u_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{\boldsymbol{j}} d_{\boldsymbol{n}\boldsymbol{j}} \phi_{\boldsymbol{n}\boldsymbol{j}}(\boldsymbol{x}), \tag{20}$$

where, for some $q \geq 1$ or $q = \infty$,

$$\begin{aligned} &\phi_{\boldsymbol{n}\boldsymbol{j}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q}\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j}), \\ &\phi(\boldsymbol{x}) = \textstyle\prod_{j=1}^d \chi_{[0,1]}(x_j), \end{aligned} \tag{21}$$

with $\chi_{[0,1]}(x)$ the characteristic function on the unit interval.

Thus, with $u_n \in V_n$ the piecewise constant approximation on Ω_n of the function $u \in C^{1,1,1}(\Omega)$, we make the hierarchical decomposition of the form $V_n = \bigoplus_{k \le n} W_k$, and write

$$u_n = \sum_{k \le n} w_k , \quad w_k \in W_k , \tag{22}$$

where

$$w_k(x) = \sum_j c_{k,j} \psi_{k,j}(x) , \tag{23}$$

with $c_{k,j} = 0$ for all j with $\|j\|$ even. In practice these coefficients $c_{k,j}$ are computed as *hierarchical surplus*, by taking the difference between the value $u(jh_k)$ and the interpolant from coarser grids. Some error bounds are found in [2].

Estimates for pw linear approximation

We approximate $u \in C^{2,2,2}(\Omega)$ by $u_n \in V_n$, where V_n denotes the space of piecewise d-linear functions on Ω_n. We take u_n such that $u_n(x) = u(x)$ for all $x \in \Omega_n^+$. We can write

$$u_n(x) = \sum_j d_{nj} \phi_{nj}(x) , \tag{24}$$

where, for some $q \ge 1$ or $q = \infty$,

$$\begin{aligned} &\phi_{nj}(x) = 2^{|n|/q} \phi(2^n x - j) , \\ &\phi(x) = \Lambda(x_1) \cdots \Lambda(x_d) , \quad \text{with } \Lambda(x) = \max(0, 1 - |x|) , \\ &\text{is the } d\text{-linear finite element type basis function.} \end{aligned} \tag{25}$$

With $u_n \in V_n$ the piecewise linear approximation on Ω_n of the function $u \in C^{2,2,2}(\Omega)$, we make a hierarchical decomposition $V_n = \bigoplus_{k \le n} W_k$, of the form (22) and (23).

The hierarchical surplus is most conveniently formulated by introducing stencil notation. Therefore, we introduce the *difference operator*

$$\Delta_h u(z) = u(z + h) - u(z) , \tag{26}$$

and the usual central difference approximation for the second derivative by *stencil* notation, as

$$\left[\frac{1}{2}, -1, \frac{1}{2}\right]_{h_j e_j} u(z) = \frac{1}{2} \Delta^2_{h_j e_j} u(z - h_j e_j) .$$

With this notation we conveniently write an expression for the hierarchical coefficients in a piecewise linear approximation. We see that d-linear interpolation leads to the following expression for the hierarchical surplus:

$$c_{\boldsymbol{k},\boldsymbol{j}} = \|\boldsymbol{h_k}\|^{1/q} \prod_{j=1,d} \left[-\frac{1}{2}, 1, -\frac{1}{2}\right]_{h_j \boldsymbol{e}_j} u(\boldsymbol{j}\boldsymbol{h_k}) .$$

Notice that the factor $\|\boldsymbol{h_k}\|^{1/q}$ cancels the scaling factor $2^{|\boldsymbol{k}|q}$ in the definition of $\phi_{\boldsymbol{k},\boldsymbol{j}}$. An expression for this coefficient $c_{\boldsymbol{k},\boldsymbol{j}}$ is found in the following lemma.

Lemma 4.1
With $l_i(x) = \min(0, -(2^{-i} - |x|)/2) = \min(0, -(h_i - |x|)/2)$, we introduce

$$L_{\boldsymbol{h}}(\boldsymbol{x}) = \prod_{i=1}^{d} l_i(x_i) . \tag{27}$$

Now

$$\begin{aligned} \int_\Omega D^{2,2,2} u(\boldsymbol{x})\, L_{\boldsymbol{h}}(\boldsymbol{x})\, d\Omega &= \prod_{i=1}^d \left(-\tfrac{1}{2}\textstyle\sum_{s_i=-1,+1} \Delta_{s_i h_i \boldsymbol{e}_i}\right) u(0) \\ &= \prod_{i=1}^d \left[-\tfrac{1}{2}, 1, -\tfrac{1}{2}\right]_{h_i \boldsymbol{e}_i} u(0) . \end{aligned}$$

Proof: The proof follows by straightforward computation. □

We derive an expression for $\|\phi\|_p$, with ϕ given by (25):

$$\begin{aligned} \|\phi\|_p^p &= \int_\Omega \prod_i |\Lambda(x_i)|^p d\Omega &&= \prod_i \left(\int_{-1}^1 (1-|x|)^p\, dx_i\right) \\ &= \prod_i \left(\tfrac{2}{p+1}\, z^{p+1}\Big|_0^1\right) &&= \prod_i \left(\tfrac{2}{p+1}\right) = \left(\tfrac{2}{p+1}\right)^d . \end{aligned}$$

So that we have

$$\|\phi\|_p = \left(\frac{2}{p+1}\right)^{d/p} . \tag{28}$$

Further, in (25) we have $\phi_{\boldsymbol{n}\boldsymbol{j}} = 2^{|\boldsymbol{n}|/q}\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j})$, and

$$\begin{aligned} \|\phi_{\boldsymbol{n}\boldsymbol{j}}\|_p^p &= \int |2^{|\boldsymbol{n}|/q}\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j})|^p\, d\Omega \\ &= 2^{|\boldsymbol{n}|p/q} \int |\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j})|^p\, d(2^{|\boldsymbol{n}|}\boldsymbol{x})\ \ 2^{-|\boldsymbol{n}|} \\ &= 2^{|\boldsymbol{n}|(p/q-1)}\ \|\phi\|^p , \end{aligned}$$

So that

$$\|\phi_{\boldsymbol{n}\boldsymbol{j}}\|_p = 2^{|\boldsymbol{n}|(1/q-1/p)}\, \|\phi\| . \tag{29}$$

This means that the norm $\|\phi_{nj}\|_p$ is independent of the level n iff we take $q = p$.

Now we compute an expression for $\|L_h(x)\|_p$, in particular for $p = 1, 2, \infty$.

$$\begin{aligned}\|L_h(x)\|_p^p &= \int \prod_{i=1}^d |l_i|^p \, d\Omega &&= 2^{d(1-p)} \prod_{i=1}^d \int_0^{h_i} (h_j - x_i)^p \, dx_i \\ &= 2^{d(1-p)} \prod_{i=1}^d \frac{h_i^{p+1}}{p+1} &&= \frac{2^{d(1-p)}}{(p+1)^d} \, \|h\|^{p+1} .\end{aligned}$$

So that we may conclude, also considering the special case $p = \infty$,

$$\|L_h(x)\|_1 = \left\| \prod_{i=1}^d l_i(x_i) \right\|_1 = 2^{-d} \, \|h\|^2 , \tag{30}$$

$$\|L_h(x)\|_2 = \left\| \prod_{i=1}^d l_i(x_i) \right\|_2 = (2/3)^{d/2} \, \|h\|^{3/2} , \tag{31}$$

$$\|L_h(x)\|_\infty = \left\| \prod_{i=1}^d l_i(x_i) \right\|_\infty = 2^{-d} \, \|h\| . \tag{32}$$

Using these expressions and Lemma 4.1, we can derive the error estimates in the following theorem.

Theorem 4.2 Let $u_n \in V_n$ be the piecewise linear approximation on Ω_n of a function $u \in C^{2,2,2}(\Omega)$ such that $u_n(x) = u(x)$ for all $x \in \Omega_n^+$. If we make the hierarchical decomposition $V_n = \oplus_{k \le n} W_k$, and write

$$u_n = \sum_{k \le n} w_k , \quad w_k \in W_k ,$$

then we have the estimates

$$\begin{aligned}\|w_k\|_2 &\le \|D^{2,2,2}u\|_2 && \|h_k\|^2 \, (2/3)^d , \\ \|w_k\|_\infty &\le \|D^{2,2,2}u\|_\infty && \|h_k\|^2 \, 2^{-d} , \\ \|u - u_n\|_2 &\le \|D^{2,2,2}u\|_2 && \tfrac{1}{16} 2^d (\tfrac{2}{3})^d \|h_n\|^2 , \\ \|u - u_n\|_\infty &\le \|D^{2,2,2}u\|_\infty && \tfrac{1}{16} (\tfrac{2}{3})^d \|h_n\|^2 .\end{aligned}$$

Proof: Using Lemma 4.1 we can obtain estimates for the hierarchical coefficients $c_{k,j}$. We fix k and we derive, writing $h := h_k$,

$$\begin{aligned}\|h\|^{-1/q} \, |c_{k,j}| &= \int_\Omega D^{2,2,2}u(x) \, \chi_{k,j}(x) \, L_h(x - jh) \, dx \\ &\le \|D^{2,2,2}u \, \chi_{k,j}\|_\infty \, 2^{-d} \|h\|^2 ,\end{aligned}$$

where $\chi_{\boldsymbol{k},\boldsymbol{j}}$ is the characteristic function for the support of $L_{\boldsymbol{h}}(\boldsymbol{x}-\boldsymbol{j}\boldsymbol{h})$, or similarly

$$\begin{aligned} \|\boldsymbol{h}\|^{-1/q}\,|c_{\boldsymbol{k},\boldsymbol{j}}| &= \int_\Omega D^{2,2,2}u(\boldsymbol{x})\,\chi_{\boldsymbol{k},\boldsymbol{j}}\,L_{\boldsymbol{h}}(\boldsymbol{x}-\boldsymbol{j}\boldsymbol{h})\,d\boldsymbol{x} \\ &\le \|D^{2,2,2}u\,\chi_{\boldsymbol{k},\boldsymbol{j}}\|_2\,(2/3)^{d/2}\|\boldsymbol{h}\|^{3/2}. \end{aligned}$$

We write $w_{\boldsymbol{k}} = \sum_{\boldsymbol{j}} c_{\boldsymbol{k},\boldsymbol{j}}\phi_{\boldsymbol{k},\boldsymbol{j}}$ with $\|\boldsymbol{j}\|$ odd, and we know that these functions $\{\phi_{\boldsymbol{k},\boldsymbol{j}}\}_{\boldsymbol{j}}$, for fixed $\boldsymbol{k}$, have disjoint supports. Hence, for the hierarchical contribution

$$\begin{aligned} \|w_{\boldsymbol{k}}\|_2^2 &= \|\boldsymbol{h}\|^{2/q}\ \ \|\textstyle\sum_{\boldsymbol{j}} c_{\boldsymbol{k},\boldsymbol{j}}\phi_{\boldsymbol{k},\boldsymbol{j}}\|_2^2 \\ &\le \|\boldsymbol{h}\|^{2/q}\ \ \textstyle\sum_{\boldsymbol{j}} \|L_{\boldsymbol{h}}\|_2^2\,\|D^{2,2,2}u\chi_{\boldsymbol{k},\boldsymbol{j}}\|_2^2\,\|\phi_{\boldsymbol{k},\boldsymbol{j}}\|^2 \\ &\le \|\boldsymbol{h}\|^{2/q}\ \ \textstyle\sum_{\boldsymbol{j}}(2/3)^d\|\boldsymbol{h}\|^3\,\|D^{2,2,2}u\chi_{\boldsymbol{k},\boldsymbol{j}}\|_2^2\,\|\boldsymbol{h}\|^{1-2/q}(2/3)^d \\ &\le \|\boldsymbol{h}\|^4\ \ \ (2/3)^d\,\|D^{2,2,2}u\|_2^2. \end{aligned}$$

For the other norm

$$\begin{aligned} \|w_{\boldsymbol{k}}\|_\infty &= \|\textstyle\sum_{\boldsymbol{j}} c_{\boldsymbol{k},\boldsymbol{j}}\phi_{\boldsymbol{k},\boldsymbol{j}}\|_\infty \\ &\le \max_{\boldsymbol{j}} \|D^{2,2,2}u\,\chi_{\boldsymbol{k},\boldsymbol{j}}\|_\infty\,\|L_{\boldsymbol{h}}\|_1\,\|\boldsymbol{h}\|^{1/q}\,\|\boldsymbol{h}\|^{-1/q}\cdot 1 \\ &= \|D^{2,2,2}u\|_\infty\,2^{-d}\|\boldsymbol{h}\|^2 \end{aligned}$$

For the error, for $p=2$ or $p=\infty$,

$$\begin{aligned} \|u-u_{\boldsymbol{n}}\|_p &= \|\textstyle\sum_{\boldsymbol{k}} w_{\boldsymbol{k}} - \sum_{\boldsymbol{k}\le\boldsymbol{n}} w_{\boldsymbol{k}}\|_p \\ &\le \textstyle\sum_{\boldsymbol{k}\not\le\boldsymbol{n}} \|w_{\boldsymbol{k}}\|_p \\ &\le C_p\,\|D^{2,2,2}u\|_p\,\textstyle\sum_{\boldsymbol{k}\not\le\boldsymbol{n}}\|\boldsymbol{h}_{\boldsymbol{k}}\|^2, \end{aligned}$$

with $C_2 = (2/3)^{d/2}$ and $C_\infty = 2^{-d}$. This yields the last two estimates, by taking into account that

$$\begin{aligned} &\textstyle\sum_{\boldsymbol{k}\not\le\boldsymbol{n}}\|\boldsymbol{h}_{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k}}\|\boldsymbol{h}_{\boldsymbol{k}}\|^2 - \sum_{\boldsymbol{k}\le\boldsymbol{n}}\|\boldsymbol{h}_{\boldsymbol{k}}\|^2 = \\ &= \textstyle\prod_{j=1}^d \sum_{k_j} 4^{-k_j} - \prod_{j=1}^d \sum_{k_j\le n_j} 4^{-k_j} \\ &\le \tfrac{1}{4}\left(\tfrac{4}{3}\right)^d \textstyle\sum_{j=1}^d (1/4)^{n_j} \\ &= \tfrac{1}{16}\left(\tfrac{4}{3}\right)^d \|\boldsymbol{h}\|^2. \end{aligned}$$

□

Approximation on sparse grids

Theorem 4.3 Let $\hat{R}_n u$ be the *piecewise constant* approximation of a function $u \in C^{1,1,1}(\Omega)$ on a sparse grid on level n:

$$\hat{R}_n u = \sum_{|\boldsymbol{k}|\leq n} w_{\boldsymbol{k}}, \quad w_{\boldsymbol{k}} \in W_{\boldsymbol{k}}, \tag{33}$$

then we have the estimate

$$\|u - \hat{R}_n u\|_p \leq C\|D^{1,1,1}u\|_p \|\boldsymbol{h}\| \log^{d-1} \|\boldsymbol{h}\|. \tag{34}$$

Proof:

$$\begin{aligned} \|u - \hat{R}_n u\|_p &\leq \textstyle\sum_{|\boldsymbol{k}|>n} \|w_{\boldsymbol{k}}\|_p \\ &= \textstyle\sum_{|\boldsymbol{k}|>n} \|\prod_{j=1}^{d} (R_{\boldsymbol{k}} - R_{\boldsymbol{k}-e_j})u\|_p \\ &\leq C\,\|D^{1,1,1}u\|_p \textstyle\sum_{|\boldsymbol{k}|>n} \|\boldsymbol{h}\| \\ &\leq C\,\|D^{1,1,1}u\|_p \textstyle\sum_{l>n} 2^{-l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \\ &\leq C\,\|D^{1,1,1}u\|_p\, G(2,n,d) \\ &\leq C\,\|D^{1,1,1}u\|_p\, C_{1nd} \frac{2^{-n} n^{d-1}}{(d-1)!} \\ &\leq C\,\|D^{1,1,1}u\|_p\, \|\boldsymbol{h}\| \log^{d-1} \|\boldsymbol{h}\|. \end{aligned}$$

Where the constants C_{1nd} are reasonably small numbers* that tend to one for $n \to \infty$. □

Theorem 4.4 Let $\hat{R}_n u$ be the *piecewise d-linear* approximation of a function $u \in C^{2,2,2}(\Omega)$ on a sparse grid on level n, as in Theorem 4.3, then, for $p = 2$ or $p = \infty$, we have the estimates

$$\|u - \hat{R}_n u\|_p \leq C\|D^{2,2,2}u\|_p \|\boldsymbol{h}\|^2 \log^{d-1} \|\boldsymbol{h}\|^{-1}. \tag{35}$$

*The value of the constant $C_{1nd} = (d-1)!\, n^{1-d} 2^n G(2,n,d)$

	n=1	n=2	n=3	n=4	n=5	$n=\infty$
d=1	1	1	1	1	1	1
d=2	4	5/2	2	7/4	8/5	1
d=3	22	8	44/9	29/8	74/25	1

Proof:

$$\begin{aligned}
\|u - \hat{R}_n u\|_p &= \|\sum_{\boldsymbol{k}} w_{\boldsymbol{k}} - \sum_{|\boldsymbol{k}|\le n} w_{\boldsymbol{k}}\|_p \\
&= \|\sum_{|\boldsymbol{k}|>n} w_{\boldsymbol{k}}\|_p \\
&\le \sum_{|\boldsymbol{k}|>n} C_p \|D^{2,2,2}u\|_p \|\boldsymbol{h}_{\boldsymbol{k}}\|^2 \\
&= C_p \|D^{2,2,2}u\|_p \sum_{l>n} \|\boldsymbol{h}_{\boldsymbol{k}}\|^2 \sum_{|\boldsymbol{k}|=l} 1 \\
&= C_p \|D^{2,2,2}u\|_p \sum_{l>n} 2^{-2l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix}.
\end{aligned}$$

Using $\sum_{|\boldsymbol{k}|=l} 1 = \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix}$ and $\sum_{|\boldsymbol{k}|\le l} 1 = \begin{pmatrix} l+d \\ d \end{pmatrix}$, we know

$$\begin{aligned}
&\sum_{l>n} 2^{-2l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \\
= \;& 2^{-2(n+1)} \cdot \begin{pmatrix} n+d \\ d-1 \end{pmatrix} \; {}_2F_1([1, 1+n+d], [2+n], 1/4) \\
= \;& G(4, n, d)
\end{aligned}$$

where ${}_2F_1$ is the generalised hypergeometric function, also known as Barnes's extended hypergeometric function. It follows that

$$G(4, n, d) \sim \frac{n^{d-1} 2^{-2n}}{3(d-1)!} \quad \text{for } n \to \infty ,$$

where the asymptotic value is reached soon for small values of d. Hence†

$$\begin{aligned}
\|u - \hat{R}_n u\|_p &\le C_p \|D^{2,2,2}u\|_p \; G(4, n, d) \\
&\le C_p \|D^{2,2,2}u\|_p \; C_{2nd} \frac{n^{d-1} 2^{-2n}}{3(d-1)!}
\end{aligned}$$

where C_{nd} is a constant that tends to one. So, we conclude that

$$\|u - \hat{R}_n u\|_2 \le \|D^{2,2,2}u\|_2 \left(\frac{2}{3}\right)^{d/2} \frac{C}{3(d-1)!} n^{d-1} 4^{-n} ,$$

$$\|u - \hat{R}_n u\|_\infty \le \|D^{2,2,2}u\|_\infty 2^{-d} \frac{C}{3(d-1)!} n^{d-1} 4^{-n} ,$$

where C is a constant that approaches unity for larger values of n. With $\|\boldsymbol{h}\| = 2^{-n}$, this proves the theorem. □

†The value of the constant $C_{2nd} = 3(d-1)! \, n^{1-d} 2^{2n} G(4, n, d)$

	n=1	n=2	n=3	n=4	n=5	$n = \infty$
d=1	1	1	1	1	1	1
d=2	10/3	13/6	16/9	19/12	22/15	1
d=3	134/9	53/9	308/81	211/71	554/235	1

Theorem 4.5 Let $\hat{R}_n u$ be the piecewise d-linear approximation of a function $u \in C^{2,1,1} \cap C^{1,2,1} \cap C^{1,1,2}(\Omega)$ on a sparse grid on level n, as in Theorem 4.3, then, for $p = 2$ and $p = \infty$, we have the estimates

$$\|u - \hat{R}_n u\|_p \leq C\|(D^{2,1,1} + D^{1,2,1} + D^{1,1,2})u\|_p \, \|h\| \log \|h\|^{-1} . \tag{36}$$

Proof: We prove, more generally, for some $m = (m_1, \cdots, m_d)$ with $1 \leq m_1, \cdots, m_d \leq 2$

$$\begin{aligned} \|u - \hat{R}_n u\|_p &\leq \sum_{|k|>n} \|w_k\|_p \\ &= \sum_{|k|>n} \| \prod_{j=1}^d (R_k - R_{k-e_j}) u\|_p \\ &\leq \sum_{|k|>n} \left(\prod_{j=1}^d h_{k_j}^{m_j}\right) C \|D^m u\|_p \\ &\leq C \|D^m u\|_p \sum_{l>n} \|h\|^2 \sum_{|k|=l} \prod_{j=1}^d 2^{k_j(2-m_j)} \\ &\leq C \|D^m u\|_p \, n^{2d-1-|m|} \, 2^{-n} . \end{aligned} \tag{37}$$

Hence, for $e \leq m < 2e$ we have

$$\|u - \hat{R}_n u\|_p \leq C\|D^m u\|_p \, \|h\| \, \log^{2d-1-|m|} \|h\|^{-1} .$$

Moreover, (37) yields, for $m = 2e$,

$$\|u - \hat{R}_n u\|_p \leq C \|D^{2,2,2} u\|_p \, n^{d-1} \, 4^{-n}$$

and hence

$$\|u - \hat{R}_n u\|_p \leq C\|D^{2,2,2} u\|_p \, \|h\|^2 \, \log^{d-1} \|h\|^{-1} .$$

□

Theorem 4.6 Let $\hat{R}_n u$ be the piecewise d-linear approximation of a function $u \in C^{2,1,1} \cap C^{1,2,1} \cap C^{1,1,2}(\Omega)$ on a sparse grid on level n, as in Theorem 4.3, then we have the estimates

$$\|u - \hat{R}_n u\|_{W_p^1} \leq C\|(D^{2,1,1} + D^{1,2,1} + D^{1,1,2})u\|_p \|h\| \, \log^{d-1} \|h\|^{-1} . \tag{38}$$

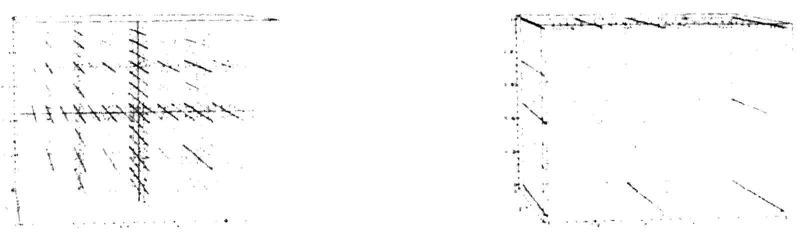
If, moreover, we know $u \in C^{2,2,2}$, then

$$\|u - \hat{R}_n u\|_{W_p^1} \leq C\|D^{2,2,2} u\|_p \, \|h\| . \tag{39}$$

Proof: Part 1:

$$\begin{aligned} \|D^\alpha w_k\|_p &= \|D^\alpha \prod_{j=1}^d (R_k - R_{k-e_j}) u\|_p \\ &\leq C^d \|h_k\| \|D^\alpha D^{1,1,1} u\|_p \end{aligned}$$

$$\begin{array}{rcl} \|D^\alpha(u-\hat{R}_n u)\|_p & \leq & \sum_{|\boldsymbol{k}|>n} C^d \,\|\boldsymbol{h_k}\|\, \|D^\alpha D^{1,1,1} u\|_p \\ & \leq & C^d \|D^\alpha D^{1,1,1} u\|_p \sum_{|\boldsymbol{k}|>n} \|\boldsymbol{h_k}\| \\ & \leq & C^d \|D^\alpha D^{1,1,1} u\|_p n^{d-1} 2^{-n} \\ & \leq & C^d \|D^\alpha D^{1,1,1} u\|_p \,\|\boldsymbol{h}\|\, \log^{d-1} \|\boldsymbol{h}\| \,. \end{array}$$



Cell centers Cell vertices

Figure 5: A sparse grid, level $n = 6$.

Part 2:

$$\|D^\alpha(u-\hat{R}_n u)\|_p \;\leq\; \sum_{|\boldsymbol{k}|>n} C\, \|\boldsymbol{h_k}\|^2 h_{k_\alpha}^{-1} \|D^{2,2,2} u\|_p$$

$$\begin{array}{rcl} \|\left(\sum_j D^{\boldsymbol{e}_j}\right)(u-\hat{R}_n u)\|_p & \leq & \|D^{2,2,2}u\|_p \sum_{|\boldsymbol{k}|>n} \|\boldsymbol{h_k}\|^2 \sum_j h_{k_j}^{-1} \\ & \leq & \|D^{2,2,2}u\|_p \, D(d,n)\,, \end{array}$$

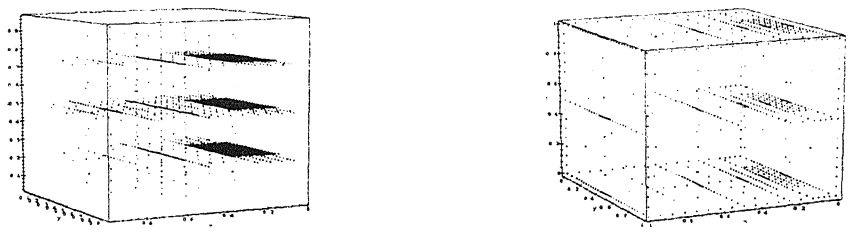
where we see that $D(d,n)$ is a number depending on n and on the dimension d. A simple computation shows

$$\begin{array}{rcl} D(1,n) & = & 2^{-n}\,, \\ D(2,n) & = & 4\; 2^{-n} - 2/3\; 4^{-n}\,, \\ D(3,n) & = & 12\; 2^{-n} - 13/3\; 4^{-n} - n\; 4^{-n}\,. \end{array}$$

So it follows that

$$\|u - \hat{R}_n u\|_{W_p^1} \leq C\, \|D^{2,2,2}u\|_p \,\|\boldsymbol{h}\|\,.$$

□



Cell centers Cell vertices

Figure 6: An adaptive grid,
$f(x, y, z) = \cos(\pi x/2)^8 \sin(\pi y)^6 z$; $\epsilon = 0.01$.

References

[1] A. Brandt. Multi-level adaptive solutions to boundary value problems. *Math. Comput.*, 31:333–390, 1977.

[2] P.W. Hemker. Sparse-grid finite-volume multigrid for 3D-problems. *Advances in Computational Mathematics*, 1995.

[3] P.W. Hemker and P. Wesseling, editors. *Multigrid methods IV*, volume 116 of *International Series of Numerical Mathematics*, Basel, 1994. Birkhäuser Verlag. Proceedings of the Fourth European Multigrid Conference, held in Amsterdam, July 6-9, 1993.

[4] C. Zenger. Sparse grids. In W. Hackbusch, editor, *Parallel Algorithms for PDE, Proc. 6th GAMM Seminar, Kiel 1990*, pages 241–251, Braunschweig, 1991. Vieweg.

Centre for Mathematics and Computer Science,
P.O.Box 94079, NL-1090 GB, Amsterdam, The Netherlands,
e-mail: P.W.Hemker@cwi.nl